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On certain definite integrals which arise in automorphic Lie theory

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Abstract. We calculate in closed form a family of definite integrals $I_n(b, a; c)$, $n = 0, 1, 2, 3, \ldots$, which arise in the calculation of regularized functional determinants associated with a compact space form X of a rank 1 Riemannian symmetric space. In the special case when $n = 0$, $b = \frac{1}{2}$, $a = \pi$, $c = 1$, and X is a Riemann surface, the integral $I_0(\frac{1}{2}, \pi; 1)$ is known, for example in the context of Polyakov string theory.

1. Introduction

Let Δ_Γ be the projection of the Laplace–Beltrami operator $\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ to a compact Riemann surface X_Γ with fundamental group Γ and genus $g \geqslant 2$. Here X_Γ is represented as the orbit space $\Gamma \setminus \pi^+$ of the upper $\frac{1}{2}$-plane π^+. For various elliptic differential operators D, and psuedo differential operators, it is important in mathematics and physics to calculate the regularized functional determinant det D. For Polyakov string theory, for example, one has the following useful result (see [7, 9]).

Theorem (1.1). Let Z_Γ be the Selberg zeta function attached to X_Γ and to the trivial representation of Γ [8]. Then $\det(-\Delta_\Gamma + s(s-1)) = Z_\Gamma(s)\Xi(s)^{2-2g}$ where the function Ξ is given as follows. For $\Gamma(\cdot)$ the usual gamma function and $\Gamma_2(\cdot)$ the Barnes double gamma function (see (2.1))

$$\Xi(s) = e^{-f+s(s-1)} \frac{\Gamma(s)}{(2\pi)^s \Gamma_2(s)^2} \tag{1.2}$$

where

$$f \stackrel{\text{def}}{=} -\log 2\pi - \frac{\pi}{4} \int_{-\infty}^{\infty} (\operatorname{sech}^2 \pi t)(\tfrac{1}{4} + t^2)[\log(\tfrac{1}{4} + t^2) - 1]\,dt. \tag{1.3}$$

Also compare [2–4, 6].

The interesting integral in (1.3) can be evaluated in terms of the derivative ζ' of Riemann's zeta function ζ at the point -1 [9]:

$$f = -\tfrac{1}{4} - \tfrac{1}{2}\log 2\pi + 2\zeta'(-1). \tag{1.4}$$

As $\zeta'(-1) = -(0.165\,421\,145)$ the value of f is $-(1.499\,54)$; compare Fried's remarks in the appendix of [4]. In this paper we consider more generally integrals of the form

$$I_n(b, a; c) = \int_{-\infty}^{\infty} t^{2n} (\operatorname{sech}^2 at)(b^2 + t^2)[\log(b^2 + t^2) - c]\,dt \tag{1.5}$$

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where $b, a > 0$, c is a complex number, and $n = 0, 1, 2, 3, \ldots$. In the main result, theorem (4.2), we give a closed form expresssion of the integrals I_n. One obtains, in particular, another proof of (1.4) by taking $n = 0, b = \frac{1}{2}, a = \pi, c = 1$. By theorem (4.2) the functions $b \to I_n(b, a; c)$, originally defined for $b > 0$, admit an explicit homomorphic continuation in terms of a new class of functions which we denote by Δ_n. The functions Δ_n satisfy a functional equation $b \to -b$ which for $n = 1$ is precisely the classical *reflection formula* for the double gamma function Γ_2; see theorems (2.16) and (2.18).

As shown in [10], given theorem (4.2) one can formulate and prove a considerably more general version of theorem (1.1) where X_Γ is replaced by a compact space form of a rank 1 Riemannian symmetric space. This is the general setting in which the integrals I_n first arise. Z_Γ makes sense in this generality and from the functional equation for Δ_n (which implies a functional equation $b \to -b$ for the integrals I_n) one can derive a new proof of the functional equation of Z_Γ [10]. Besides the mathematical applications just pointed out, the evaluation of the integrals I_n, especially in case $n = 0$, is of interest for physical reasons in connection with multiloop calculations for fermonic string theory and random surfaces, as D'Hoker and Phong point out in [2]. The determinant in theorem (1.1) and, more generally, the determinant of Laplacians acting on arbitrary tensor and spinor fields arise from quantum fluctuations and Faddeev–Popov gauge fixing [6].

In the route toward proving the main result we first compute the integrals

$$J_n(b, a) = \int_0^\infty t^{2n}(\operatorname{sech}^2 at) \log(b^2 + t^2)\, dt \tag{1.6}$$

for $b, a > 0$, $n = 0, 1, 2, \ldots$; see theorem (3.19) wherein occurs the term $n\zeta'(1 - 2n)$ involving the special value of the derivative of the Riemann zeta function. In the Riemann surface case one needs the J_n only for $n = 0, 1$ and hence one only encounters the special value $\zeta'(-1)$. Compare equation (1.4).

Although the integrals I_n, J_n arose initially by way of certain Lie theoretical and mathematical physics considerations, as we have indicated, the present paper requires no knowledge of Lie theory nor of Selberg's zeta function Z_Γ.

2. The functions Δ_n

We introduce the functions Δ_n which play a key role in this paper. As in the introduction Γ will denote the usual gamma function and Γ_2 will denote the Barnes double gamma function [1] defined by

$$\frac{1}{\Gamma_2(s+1)} = (2\pi)^{s/2} \exp\left[-\frac{s}{2} - \left(\frac{\gamma + 1}{2}\right)s^2\right] \prod_{k=1}^{\infty} \left(1 + \frac{s}{k}\right)^k \exp\left(-s + \frac{s^2}{2k}\right) \tag{2.1}$$

where

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\right) \tag{2.2}$$

is Euler's constant and $s \in \mathbb{C}$, the field of complex numbers. $1/\Gamma_2$ is an entire function whose zeros are the points $0, -1, -2, -3, \ldots$. For $r \in \mathbb{R}$, the field of real numbers, we define

$$U_r = \mathbb{C} - \{x + \mathrm{i}y | y = 0 \text{ and } x \leqslant -r\}. \tag{2.3}$$

U_r is therefore an open simply connected domain in $\mathbb{C}$. We shall choose the principal branch of the complex logarithm. Thus $\log\Gamma$ and $\log 1/\Gamma_2$ are holomorphic functions on U_0. For $n = 1, 2, 3, \ldots$ we define ϕ_n to be the unique holomorphic primitive of $Z^n \log\Gamma(Z+1)$ on U_1 which vanishes at 0:

$$\phi_n'(Z) = Z^n \log\Gamma(Z+1) \text{ on } U_1 \qquad \phi_n(0) = 0. \tag{2.4}$$

With the preceding definitions in place we introduce the well-defined holomorphic functions Δ_n on $U_{\pi/2}$: For $Z \in U_{\pi/2}$, $n = 1, 2, 3, \ldots$

$$\begin{aligned}\Delta_n(Z) &\stackrel{\text{def}}{=} \left[-2n\pi^2 Z^{2n-1} + \frac{n(2n-1)\pi^{2n}}{2^{2n-3}} Z - \frac{n(2n-1)\pi^{2n+1}}{2^{2n-2}}\right] \log\Gamma\left(\frac{Z}{\pi} + \frac{1}{2}\right) \\ &\quad + \frac{n(2n-1)\pi^{2n+1}}{2^{2n-3}} \left[\frac{1}{2}\left(\frac{Z}{\pi} - \frac{1}{2}\right)\log 2\pi - \frac{1}{2}\left[\left(\frac{Z}{\pi} - \frac{1}{2}\right)\left(\frac{Z}{\pi} + \frac{1}{2}\right)\right]\right. \\ &\quad \left. - \log\frac{1}{\Gamma_2}\left(\frac{Z}{\pi} + \frac{1}{2}\right)\right] + \frac{n(2n-1)}{2^{2n-3}} \sum_{j=1}^{2n-2} \binom{2n-2}{j} 2^j \phi_j\left(\frac{Z}{\pi} - \frac{1}{2}\right)\end{aligned} \tag{2.5}$$

we set $\Delta_0 = 0$. Since $\phi_n(0) = 0$ in (2.4) we have

$$\Delta_n(\tfrac{1}{2}\pi) = 0 \qquad \text{for all } n. \tag{2.6}$$

Note also that for $n = 1$

$$\begin{aligned}\Delta_1(Z) &= -\pi^3 \log\Gamma\left(\frac{Z}{\pi} + \frac{1}{2}\right) + 2\pi^3 \left[\frac{1}{2}\left(\frac{Z}{\pi} - \frac{1}{2}\right)\log 2\pi\right. \\ &\quad \left. - \frac{1}{2}\left[\left(\frac{Z}{\pi} - \frac{1}{2}\right)\left(\frac{Z}{\pi} + \frac{1}{2}\right)\right] + \log\Gamma_2\left(\frac{Z}{\pi} + \frac{1}{2}\right)\right].\end{aligned} \tag{2.7}$$

Let

$$\psi = \Gamma'/\Gamma \tag{2.8}$$

which is meromorphic with simple poles at $Z = 0, -1, -2, -3, \ldots$.

The function Δ_n arises as the solution of a simple differential equation that we shall need. Namely

Proposition (2.9). $\Delta_n'(Z) = -2n\pi Z^{2n-1}\psi(Z/\pi + \frac{1}{2})$ on $U_{\pi/2}$ for $n \geqslant 1$.

Proof. For $h^{\pm}(Z) \stackrel{\text{def}}{=} Z/\pi \pm \frac{1}{2}$, $L \stackrel{\text{def}}{=} \log\Gamma$, $L_2 = \log 1/\Gamma_2$, direct differentiation of (2.5) yields

$$\begin{aligned}\Delta_n'(Z) &\stackrel{(i)}{=} \left[-2n\pi^2 Z^{2n-1} + \frac{n(2n-1)\pi^{2n} Z}{2^{2n-3}} - \frac{n(2n-1)\pi^{2n+1}}{2^{2n-2}}\right] \frac{L'(h^+(Z))}{\pi} \\ &\quad + L(h^+(Z))\left[-2n\pi^2(2n-1)Z^{2n-2} + \frac{n(2n-1)\pi^{2n}}{2^{2n-3}}\right] \\ &\quad + \frac{n(2n-1)}{2^{2n-3}}\pi^{2n+1}\left[\frac{\log 2\pi}{2\pi} - \frac{Z}{\pi^2} - L_2'(h^+(Z))/\pi\right] \\ &\quad + \frac{n(2n-1)\pi^{2n+1}}{2^{2n-3} = 2^{2n-2}2^{-1}} \sum_{j=1}^{2n-2} \binom{2n-2}{j} 2^j \phi_j'(h^-(Z)) \frac{1}{\pi}\end{aligned}$$

where the latter term in (i) coincides with the negative of the second term in (i). Namely, by definition (2.4), and the binomial theorem this latter term is

$$2n(2n-1)\pi^{2n+1}\sum_{j=1}^{2n-2}\binom{2n-2}{j}\left(\frac{1}{2}\right)^{2n-2-j}\frac{h^-(Z)^j}{\pi}L(h^+(Z))$$
$$=2n(2n-1)\pi^{2n+1}\frac{L(h^+(Z))}{\pi}\left[\left(\frac{1}{2}+h^-(Z)\right)^{2n-2}-\binom{2n-2}{0}\left(\frac{1}{2}\right)^{2n-2}\right]$$
$$=2n(2n-1)\pi^{2n}L(h^+(Z))\left[\left(\frac{Z}{\pi}\right)^{2n-2}-\frac{1}{2^{2n-2}}\right].$$

(i) therefore simplifies to

$$\Delta_n'(Z)\overset{\text{(ii)}}{=}\left[-2n\pi^2Z^{2n-1}+\frac{n(2n-1)\pi^{2n}Z}{2^{2n-3}}-\frac{n(2n-1)\pi^{2n+1}}{2^{2n-2}}\right]\frac{\psi(h^+(Z))}{\pi}$$
$$+\frac{n(2n-1)\pi^{2n+1}}{2^{2n-3}}\left[\frac{\log 2\pi}{2\pi}-\frac{Z}{\pi^2}-\frac{1}{\pi}L_2'(h^+(Z))\right]$$

by definition (2.8). Next we use that for $B\overset{\text{def}}{=}1/\Gamma_2$

$$\frac{B'(Z+1)}{B(Z+1)}=\tfrac{1}{2}\log 2\pi+\tfrac{1}{2}-Z+Z\psi(Z) \tag{2.10}$$

with $Z\psi(Z)+1=Z\psi(Z+1)$ for $Z\neq-1,-2,-3,\ldots$; cf p 661, formula (4) of [5]. Note that as $Z\doteq 0$ is a simple pole of ψ with residue $=-1$, the function $Z\to Z\psi(Z)$ defined to be -1 at $Z=0$ has $Z=0$ as a removable singularity. For $Z\in U_{\pi/2}$, $Z/\pi-\frac{1}{2}\neq-1,-2,-3,\ldots$. Also by definition of L_2, $L_2'=B'/B$. Thus we may choose Z in (2.10) as $Z/\pi-\frac{1}{2}$ for $Z\in U_{\pi/2}$ to obtain

$$\frac{1}{\pi}L_2'(h^+(Z))=\frac{\log 2\pi}{2\pi}+\frac{1}{2\pi}-\frac{1}{\pi}\left(\frac{Z}{\pi}-\frac{1}{2}\right)-\frac{1}{\pi}+\frac{1}{\pi}\left(\frac{Z}{\pi}-\frac{1}{2}\right)\psi\left(\frac{Z}{\pi}+\frac{1}{2}\right)$$
$$=\frac{\log 2\pi}{2\pi}-\frac{Z}{\pi^2}+\frac{1}{\pi}\left(\frac{Z}{\pi}-\frac{1}{2}\right)\psi(h^+(Z))$$

by which equation (ii) reduces to

$$\Delta_n'(Z)=\left[-2n\pi Z^{2n-1}+\frac{n(2n-1)\pi^{2n-1}Z}{2^{2n-3}}-\frac{n(2n-1)\pi^{2n}}{2^{2n-2}}\right]\psi(h^+(Z))$$
$$+\frac{n(2n-1)\pi^{2n+1}}{2^{2n-3}}\left[-\frac{1}{\pi}\left(\frac{Z}{\pi}-\frac{1}{2}\right)\psi(h^+(Z))\right]$$
$$=-2n\pi Z^{2n-1}\psi(h^+(Z))$$

as desired. From the formulae

$$\Gamma(\tfrac{1}{2})=\pi^{1/2}\qquad \Gamma_2(\tfrac{1}{2})^{-1}=A^{-3/2}\pi^{-1/4}e^{1/8}2^{1/24} \tag{2.11}$$

where

$$\log A = -\zeta'(-1) + \tfrac{1}{12} \tag{2.12}$$

with ζ as in section 1, one derives for $n \geqslant 1$

$$\begin{aligned}\Delta_n(0) = &-\frac{2n(2n-1)\pi^{2n+1}}{2^{2n}}\log\pi - \frac{7}{3}\frac{n(2n-1)\pi^{2n+1}}{2^{2n}}\log 2 + \frac{n(2n-1)}{2^{2n}}\pi^{2n+1}\\ &-\frac{12n(2n-1)\pi^{2n+1}}{2^{2n}}\zeta'(-1) + \frac{8n(2n-1)\pi^{2n+1}}{2^{2n}}\sum_{j=1}^{2n-2}\binom{2n-2}{j}2^j\phi_j(-\tfrac{1}{2}).\end{aligned} \tag{2.13}$$

With definition (2.3) in mind define

$$\begin{aligned}U_{1/2}^- &= \{Z \in U_{1/2} \mid -Z \in U_{1/2}\}\\ &= \pi^+ \cup \pi^- \cup (-\tfrac{1}{2}, \tfrac{1}{2})\end{aligned} \tag{2.14}$$

where π^+, π^- denote the upper, lower $\frac{1}{2}$-plane, respectively. Then by proposition (2.9) and the relation

$$\psi(\tfrac{1}{2} + Z) = \psi(\tfrac{1}{2} - Z) + \pi\tan\pi Z \tag{2.15}$$

one obtains

Theorem (2.16). (The functional equation for Δ_n.) Let C_Z for Z in $U_{1/2}^-$ be any contour in $U_{1/2}^-$ from 0 to Z. Then for $n \geqslant 1$

$$\frac{-\Delta_n(\pi Z) + \Delta_n(-\pi Z)}{2\pi^{2n+1}} = \pi n\int_{C_Z} s^{2n-1}\tan\pi s\, ds. \tag{2.17}$$

On the other hand Δ_n is given explicitly by definition (2.5). Writing out the left-hand side of (2.17) fully and simplifying, we see that theorem (2.16) is equivalent to

Theorem (2.18). For $n \geqslant 1$, $Z \in U_{1/2}^-$, and C_Z any contour in $U_{1/2}^-$ from 0 to Z (see (2.14))

$$\begin{aligned}n\pi\int_{C_Z} s^{2n-1}\tan\pi s\, ds = &\left[nZ^{2n-1} - \frac{n(2n-1)}{2^{2n-2}}Z\right]\log\Gamma(Z+\tfrac{1}{2})\Gamma(-Z+\tfrac{1}{2})\\ &+\frac{n}{2}\frac{(2n-1)}{2^{2n-2}}\log\frac{\Gamma(Z+\frac{1}{2})}{\Gamma(-Z+\frac{1}{2})} - \frac{n(2n-1)}{2^{2n-2}}Z\log 2\pi\\ &+\frac{n(2n-1)}{2^{2n-2}}\log\frac{\Gamma_2(-Z+\frac{1}{2})}{\Gamma_2(Z+\frac{1}{2})} + \frac{n(2n-1)}{2^{2n-2}}\\ &\times\sum_{j=1}^{2n-2}\binom{2n-2}{j}2^j[\phi_j(-Z-\tfrac{1}{2}) - \phi_j(Z-\tfrac{1}{2})].\end{aligned}$$

In particular, for $n = 1$ we obtain the *reflection formula*

$$\pi\int_{C_Z} s\tan\pi s\, ds = \frac{1}{2}\log\frac{\Gamma(Z+\frac{1}{2})}{\Gamma(-Z+\frac{1}{2})} - Z\log\pi + \log\frac{\Gamma_2(Z+\frac{1}{2})}{\Gamma_2(-Z+\frac{1}{2})}. \tag{2.19}$$

Note that in theorem (2.18)

$$\Gamma(Z+\tfrac{1}{2})\Gamma(-Z+\tfrac{1}{2}) = \pi/(\cos\pi Z).$$

3. The integrals $J_n(b, a)$

The purpose of this section is to compute the integrals $J_n(b,a)$ defined in (1.6). These integrals completely determine the integrals $I_n(b,a;c)$ of (1.5).

B_n will denote the nth Bernoulli number:

$$\frac{Z}{e^Z-1}=\sum_{n=0}^{\infty}\frac{B_n Z^n}{n!}\qquad \text{for } |Z|<2\pi. \tag{3.1}$$

Let

$$\begin{aligned} I_n(a) &= \int_0^\infty x^{2n}\operatorname{sech}^2 ax\,\mathrm{d}x \\ &= \frac{2^{2n-2}}{2^{2n}}\frac{(-1)^{n+1}}{\pi}\left(\frac{\pi}{a}\right)^{2n+1}B_{2n} \end{aligned}\tag{3.2}$$

(cf [5], p 353) and let

$$f_n(a)=\int_0^\infty t^{2n}(\operatorname{sech}^2 at)\log(1+t^2)\,\mathrm{d}t \tag{3.3}$$

for $a>0$, $n=0,1,2,3,\ldots$. Using

$$\log(b^2+x^2)=2\log b+\log(1+(x/b)^2)$$

and the change of variables $t=x/b$ for $b>0$ we see that in view of (3.2), $J_n(b,a)$ is determined by the function f_n:

$$J_n(b,a)=2(\log b)I_n(a)+b^{2n+1}f_n(ab). \tag{3.4}$$

By the change of variables $at=x$

$$f_n(a)=\frac{1}{a^{2n+1}}\int_0^\infty x^{2n}(\operatorname{sech}^2 x)\log\left(1+\frac{x^2}{a^2}\right)\mathrm{d}x \tag{3.5}$$

where the integral in (3.5) clearly converges uniformly in $a\geqslant 1$. We may therefore differentiate under the integral sign in (3.5) to obtain (for $a\geqslant 1$)

$$f'(a)\frac{-2}{a^{2n+2}}\int_0^\infty\frac{x^{2n}x^2\operatorname{sech}^2 x\,\mathrm{d}x}{a^2+x^2}-\frac{(2n+1)}{a}f_n(a)=\frac{-2}{a^{2n+2}}I_n(1)+g_n(a)-\frac{(2n+1)}{a}f_n(a) \tag{3.6}$$

where

$$g_n(a)\overset{\text{def}}{=}\frac{2}{a^{2n}}\int_0^\infty\frac{x^{2n}\operatorname{sech}^2 x\,\mathrm{d}x}{a^2+x^2}. \tag{3.7}$$

On the other hand, we can write (3.5) alternatively as

$$f_n(a)=\frac{1}{a^{2n+1}}\int_0^\infty x^{2n}(\operatorname{sech}^2 x)\log(a^2+x^2)\,\mathrm{d}x-\frac{2\log a}{a^{2n+1}}I_n(1) \tag{3.8}$$

(using $\log(1+x^2/a^2)=\log(a^2+x^2)-2\log a$) where the integral in (3.8) converges uniformly in $0<a\leqslant 1$. Differentiating (3.8) under the integral sign we therefore also obtain (3.6) for $0<a\leqslant 1$. That is, (3.6) holds for $a>0$, and as a first-order linear differential equation it has a standard trivial solution:

$$f_n(a)=\frac{1}{a^{2n+1}}\left[\int a^{2n+1}\left(g_n(a)-\frac{2}{a^{2n+2}}I_n(1)\right)\mathrm{d}a+c_n'\right] \tag{3.9}$$

for $c_n'=$ constant. The integration in (3.9) can be carried out using proposition (2.9) and the following lemma.

Lemma (3.10). For $a > 0, n = 0, 1, 2, 3, \ldots$

$$\int_0^\infty \frac{t^{2n}\operatorname{sech}^2 t}{a^2+t^2}\,\mathrm{d}t = \frac{(-1)^n a^{2n-1}}{\pi}\zeta\left(2, \frac{a}{\pi}+\frac{1}{2}\right) + (-1)^{n+1}2a^{2(n-1)}\sum_{j=0}^{n-1}\frac{\pi^{2j}}{a^{2j}2^{2j}}\left(\frac{2-2^{2j}}{2}\right)B_{2j} \tag{3.11}$$

where ζ is the Hurwitz zeta function:

$$\zeta(s,\alpha) \overset{\text{def}}{=} \sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s} \qquad \text{for } \operatorname{Re}s > 1 \qquad \alpha \neq -1, -2, -3, \ldots. \tag{3.12}$$

Since $t^{2n}/(b^2+t^2) = t^{2(n-1)} - b^2t^{2(n-1)}/(b^2+t^2)$ for $n \geqslant 1$, lemma (3.10) follows by induction, using (3.2), once one knows that

$$\int_0^\infty \frac{\operatorname{sech}^2 t\,\mathrm{d}t}{b^2+t^2} = \frac{1}{\pi b}\zeta\left(2, \frac{b}{\pi}+\frac{1}{2}\right) \tag{3.13}$$

the proof of which will be remarked on later.

By definition (3.7) and equation (3.11) one has

$$\int a^{2n+1}g_n(a)\,\mathrm{d}a = \int \frac{2(-1)^n}{\pi}a^{2n}\zeta\left(2, \frac{a}{\pi}+\frac{1}{2}\right)\mathrm{d}a + 4(-1)^{n+1}\sum_{j=0}^{n-1}\frac{\pi^{2j}}{2^{2j}}\left(\frac{2-2^{2j}}{2}\right)B_{2j}\frac{a^{2n-2j}}{2n-2j}. \tag{3.14}$$

But one knows that ψ in (2.8) and ζ in (3.12) are related by

$$\psi'(s) = \zeta(2, s). \tag{3.15}$$

Integration by parts therefore yields

$$\begin{aligned}\int a^{2n}\zeta\left(2, \frac{a}{\pi}+\frac{1}{2}\right)\mathrm{d}a &= a^{2n}\pi\psi\left(\frac{a}{\pi}+\frac{1}{2}\right) - \int 2\pi n a^{2n-1}\psi\left(\frac{a}{\pi}+\frac{1}{2}\right)\mathrm{d}a \\ &= a^{2n}\pi\psi\left(\frac{a}{\pi}+\frac{1}{2}\right) + \Delta_n(a) + c_n''\end{aligned} \tag{3.16}$$

where the latter equality follows by proposition (2.9)! c_n'' is a constant of integration. From equations (3.9), (3.14) and (3.16)

$$\begin{aligned}f_n(a) = {} & \frac{2(-1)^n}{a}\psi\left(\frac{a}{\pi}+\frac{1}{2}\right) + \frac{2(-1)^n}{\pi a^{2n+1}}\Delta_n(a) + \frac{4(-1)^{n+1}}{a^{2n+1}}\sum_{j=0}^{n-1}\frac{\pi^{2j}}{2^{2j}}\left(\frac{2-2^{2j}}{2}\right) \\ & \times B_{2j}\frac{a^{2n-2j}}{2n-2j} + \frac{2(-1)^n c_n''}{\pi a^{2n+1}} + \frac{c_n'}{a^{2n+1}} - \frac{2I_n(1)\log a}{a^{2n+1}}.\end{aligned} \tag{3.17}$$

That is, by (3.2) we get

Theorem (3.18). The integral $f_n(a)$ in (3.3) is given by

$$f_n(a) = \frac{2(-1)^n}{a}\psi\left(\frac{a}{\pi}+\frac{1}{2}\right) + \frac{2(-1)^n}{\pi a^{2n+1}}\Delta_n(a) + \frac{(-1)^{n+1}}{a^{2n+1}}\sum_{j=0}^{n-1}\frac{\pi^{2j}(2-2^{2j})}{2^{2j}a^{2j}(n-j)}B_{2j}$$
$$+ \frac{2(-1)^{n+1}\pi^{2n}}{2^{2n}a^{2n+1}}(2-2^{2n})B_{2n}\log a + \frac{c_n}{a^{2n+1}}$$

where c_n = constant. We now state

Theorem (3.19). The integral $J_n(b,a)$ in (1.6) is given by

$$J_n(b,a) = \frac{b^{2n}}{a}(-1)^{n+1}\sum_{j=0}^{n-1}\frac{\pi^{2j}(2-2^{2j})}{(n-j)2^{2j}a^{2j}b^{2j}}B_{2j} + \frac{c_n}{a^{2n+1}} + \frac{2(-1)^{n+1}(2-2^{2n})}{2^{2n}a^{2n+1}}\pi^{2n}B_{2n}\log a$$
$$+ \frac{2(-1)^n b^{2n}}{a}\psi\left(\frac{ab}{\pi}+\frac{1}{2}\right) + \frac{\Delta_n(ab)2(-1)^n}{\pi a^{2n+1}}$$

(see (2.5), (2.8), (3.11)) where c_n is a constant given by

$$\frac{c_n}{\pi^{2n+1}} + \frac{2(-1)^{n+1}(2-2^{2n})}{2^{2n}\pi}B_{2n}\log\pi + \frac{2(-1)^n\Delta_n(0)}{\pi^{2n+2}}$$
$$= \frac{4(2^{2n}-2)}{2^{2n}\pi}(-1)^{n+1}\left[n\zeta'(1-2n) + \frac{B_{2n}}{4n}\right] + \frac{4(-1)^{n+1}(\log 2)}{2^{2n}\pi}B_{2n} \qquad (3.20)$$

for $n \geqslant 1$, where $\Delta_n(0)$ is given by (2.13);

$$c_0 = 2\log\pi \qquad (3.21)$$

Up to computation of the constants c_n, theorem (3.19) follows from equations (3.2), (3.4) and theorem (3.18). As a preliminary step towards finding the c_n, choose $a = \pi$ and $b = 1/m$ in the theorem (3.19), $m = 1, 2, 3, \ldots$, and let $m \to \infty$:

$$\lim_{m\to\infty}\int_0^\infty x^{2n}(\operatorname{sech}^2\pi x)\log\left(\frac{1}{m^2}+x^2\right)\mathrm{d}x$$
$$= \lim_{m\to\infty} J_n\left(\frac{1}{m},\pi\right)$$
$$= \frac{c_n}{\pi^{2n+1}} + \frac{2(-1)^{n+1}(2-2^{2n})}{2^{2n}\pi}B_{2n}\log\pi + \frac{2(-1)^n}{\pi^{2n+2}}\Delta_n(0) \qquad (3.22)$$

for $n \geqslant 1$, where one checks that the limit on the left-hand side of (3.22) can be taken under the integral sign. That is, by dominated convergence

$$2\int_0^\infty x^{2n}(\operatorname{sech}^2\pi x)\log x\,\mathrm{d}x = \frac{c_n}{\pi^{2n+1}} + \frac{2(-1)^{n+1}(2-2^{2n})}{2^{2n}\pi}B_{2n}\log\pi + \frac{2(-1)^n\Delta_n(0)}{\pi^{2n+2}} \qquad (3.23)$$

for $n \geqslant 1$, with $\Delta_n(0)$ given by (2.13).

Consider therefore the integral in (3.23). The integral $\int_1^\infty x^{s-1}\operatorname{sech}^2 ax\,dx$, for $a > 0$, converges uniformly on compact subsets of $\operatorname{Re}s > 0$. It thus defines a holomorphic function of s which may be differentiated under integral sign on $\operatorname{Re}s > 0$. A similar statement follows for $\int_0^1 = \int_1^\infty$ under the transformation $x = 1/t$. That is, $s \to I(s) \stackrel{\text{def}}{=} \int_0^\infty x^{s-1}\operatorname{sech}^2 ax\,dx$ is holomorphic on $\operatorname{Re}s > 0$ and

$$\frac{d}{ds}\int_0^\infty x^{s-1}\operatorname{sech}^2 ax\,dx = \int_0^\infty x^{s-1}(\log x)\operatorname{sech}^2 ax\,dx \tag{3.24}$$

on $\operatorname{Re}s > 0$ for $a > 0$. On the other hand, by p 352 of [5]

$$I(s) = \frac{4}{(2a)^s}(1 - 2^{2-s})\Gamma(s)\zeta(s-1) \tag{3.25}$$

for $\operatorname{Re}s > 0$, $s \neq 2$;

$$I(2) = (1/a^2)\log 2. \tag{3.26}$$

Carrying out the differentiation in (3.24) one obtains

$$\int_0^\infty x^{s-1}(\log x)\operatorname{sech}^2 ax\,dx = \frac{4}{(2a)^s}(1-2^{2-s})[\Gamma(s)\zeta'(s-1)+\zeta(s-1)\Gamma'(s)]+\Gamma(s)\zeta(s-1) \times\left[\frac{4}{(2a)^s}2^{2-s}\log 2 - \frac{(1-2^{2-s})4}{(2a)^s}\log 2a\right] \tag{3.27}$$

for $\operatorname{Re}s > 0$, $s \neq 2$, $a > 0$. In (3.27) choose $s = 1 + 2n$, $n = 0, 1, 2, 3, \ldots$, and apply the special value formula

$$\zeta(2n) = 2^{2n-1}\pi^{2n}(-1)^{n+1}B_{2n}/(2n)!. \tag{3.28}$$

By (2.8), $\Gamma'(1+2n) = \Gamma(1+2n)\psi(1+2n) = (2n)!\psi(1+2n)$. That is, (3.27) implies

Corollary (3.29). For $a > 0$, $n = 0, 1, 2, 3, \ldots$

$$\int_0^\infty x^{2n}(\log x)\operatorname{sech}^2 ax\,dx = \frac{4(1-2^{1-2n})}{(2a)^{1+2n}}[(2n)!\zeta'(2n) + 2^{2n-1}\pi^{2n}(-1)^{n+1}B_{2n}\psi(1+2n)] + 2^{2n-1}\pi^{2n}(-1)^{n+1}B_{2n}\left[\frac{4.2^{1-2n}}{(2a)^{1+2n}}\log 2 - \frac{(1-2^{1-2n})4\log 2a}{(2a)^{1+2n}}\right].$$

It is useful to rewrite corollary (3.29) by differentiating the functional equation

$$\frac{\pi^{s/2}\zeta(1-s)}{\Gamma(s/2)} = \frac{\pi^{(1-s)/2}\zeta(s)}{\Gamma((1-s)/2)} \tag{3.30}$$

for ζ to obtain

$$-\frac{\zeta'(1-s)\pi^{s/2}}{\Gamma(s/2)} = \frac{\pi^{(1-s)/2}[\zeta'(s) - \zeta(s)\log\pi]}{\Gamma((1-s)/2)} \frac{+\pi^{(1-s)/2}\zeta(s)[\psi(s/2) + \psi((1-s)/2)]}{2\Gamma((1-s)/2)} \tag{3.31}$$

and, by taking $s = 2n$ in (3.31) in conjunction with (3.28): For $n \geqslant 1$

$$-\frac{\zeta'(1-2n)\pi^n}{(n-1)!} = \frac{\pi^{\frac{1}{2}-n}\zeta'(2n)}{\Gamma(\frac{1}{2}-n)} - \frac{\pi^{\frac{1}{2}+n}(\log\pi)2^{2n-1}(-1)^{n+1}B_{2n}}{\Gamma(\frac{1}{2}-n)(2n!)} + \frac{\pi^{\frac{1}{2}+n}2^{2n-2}(-1)^{n+1}}{\Gamma(\frac{1}{2}-n)(2n)!}B_{2n}[\psi(n)+\psi(\tfrac{1}{2}-n)]. \quad (3.32)$$

As

$$\Gamma(\tfrac{1}{2}-n) = \frac{(-1)^n 2^{2n} n! \sqrt{\pi}}{(2n)!} \quad (3.33)$$

and as

$$\psi(1+2n) = \psi(2n) + \frac{1}{2n} = \frac{1}{2}\left[\psi(n) + \psi\left(\frac{1}{2}-n\right)\right] + \log 2 + \frac{1}{2n} \quad (3.34)$$

we see that

$$\zeta'(2n)(2n)! = (-1)^{n+1}\zeta'(1-2n)\pi^{2n}2^{2n}n + (-1)^{n+1}\pi^{2n}(\log 2\pi)2^{2n-1}B_{2n} + (-1)^n\pi^{2n}2^{2n-1}B_{2n}\left[\psi(1+2n) - \frac{1}{2n}\right] \quad (3.35)$$

which implies that an alternative statement of corollary (3.29) is

Corollary (3.36). For $a > 0, n = 1, 2, 3, \ldots$

$$\int_0^\infty x^{2n}(\log x)\,\mathrm{sech}^2\, ax\, \mathrm{d}x = 2\frac{(2^{2n}-2)}{2^{2n}}\frac{(-1)^{n+1}}{a}\left(\frac{\pi}{a}\right)^{2n}\left[\zeta'(1-2n)n + \frac{B_{2n}}{4n} + \frac{B_{2n}}{2}\log\frac{\pi}{a}\right] + \frac{2(\log 2)}{2^{2n}}\frac{(-1)^{n+1}}{a}\left(\frac{\pi}{a}\right)^{2n}B_{2n}.$$

The logarithmic derivative ψ is thus eliminated in this version of corollary (3.29). The value of c_n for $n \geqslant 1$ claimed in equation (3.20) now follows from equation (3.23) by taking $a = \pi$ in corollary (3.36).

c_0 is computed similarly but more simply. Namely in place of equation (3.23) one has

$$2\int_0^\infty (\mathrm{sech}^2\,\pi x)\log x\,\mathrm{d}x = \frac{c_0}{\pi} - \frac{2}{\pi}\log\pi + \frac{2}{\pi}\psi(\tfrac{1}{2}) \quad (3.37)$$

since $\Delta_0(x) = 0$ for $x > 0$ by (2.5). Thus in constrast to (3.23), the logarithmic derivative term in ψ survives (from theorem (3.19)) in case $n = 0$. By corollary (3.29) the right-hand side of (3.37) is $2(\gamma - 2\log 2)/\pi$, since $\zeta'(0) = -\frac{1}{2}\log 2\pi$, $\psi(1) = -\gamma$ (for γ in (2.2)). Also as $\psi(\frac{1}{2}) = -\gamma - 2\log 2$, equation (3.21) follows.

One piece of unfinished business remains concerning the proof of theorem (3.19). Namely we must check equation (3.13). Consider

$$\int_R \int_R |(\mathrm{sech}^2 t)e^{ity}e^{-b|y|}|\, dt\, dy = \int_R \mathrm{sech}^2 t\, dt \int_R e^{-b|y|}\, dy < \infty \tag{3.38}$$

where $dt\, dy$ denotes the Lebesgue measure on $R \times R$. By Fubini's theorem we therefore have

$$\int_R \int_R (\mathrm{sech}^2 t)e^{ity}e^{-b|y|}\, dt\, dy = \int_R \int_R (\mathrm{sech}^2 t)e^{ity}e^{-b|y|}\, dy\, dt. \tag{3.39}$$

That is, if $\hat{f}$ is the Fourier transform of an L^1-function f,

$$\hat{f}(y) \overset{\mathrm{def}}{=} \int_R e^{ity} f(t)\, dt \tag{3.40}$$

then (3.39) reads

$$\int_R e^{-b|y|}\hat{f}(y)\, dy = \int_R f(t)\hat{h}(t)\, dt \tag{3.41}$$

for $f(t) = \mathrm{sech}^2 t$, $h(t) = e^{-b|t|}$. But $\hat{h}(t) = 2b/(t^2 + b^2)$ and $\hat{f}(y) = \pi y/\sin h(\pi y/2)$. (3.41) therefore reads

$$\begin{aligned}\int_0^\infty \frac{\mathrm{sech}^2 t}{t^2 + b^2}\, dt &= \frac{1}{2b}\int_0^\infty e^{-b|y|}\frac{\pi y}{\sin h\pi y/2}\, dy \\ &= \frac{2}{\pi b}\int_0^\infty e^{-(2b/\pi)x}\frac{x}{\sin hx}\, dx\end{aligned} \tag{3.42}$$

By formula (3.552), no 1, p 361 of [5] the second integral on the right-hand side of (3.42) has the value $(1/\pi b)\zeta(2, (b/\pi) + \frac{1}{2})$, which proves (3.13). We note that equation (3.41) also follows by the general Plancherel formula for R.

4. The integrals $I_n(b, a; c)$

By definitions (1.5), (1.6) and (3.2)

$$I_n(b, a; c) = 2b^2 J_n(b, a) + 2J_{n+1}(b, a) - 2cb^2 I_n(a) - 2cI_{n+1}(a). \tag{4.1}$$

By theorem (3.19), equation (3.2), and some algebraic manipulation we therefore derive the following main theorem.

Theorem (4.2). The integral $I_n(b, a; c)$ in (1.5) is given by

$$\begin{aligned}\pi I_n(b, a; c) &= b^2\left(\frac{\pi}{a}\right)^{2n-1}\frac{(2 - 2^{2n})}{2^{2n-2}}(-1)^{n+1}B_{2n}\log a \\ &\quad - \left(\frac{\pi}{a}\right)^{2n+3}\frac{(2 - 2^{2n+2})}{2^{2n}}(-1)^{n+1}B_{2n+2}\log a + \frac{2\pi}{a}b^{2n+2}(-1)^{n+1} \\ &\quad \times \sum_{j=0}^{n-1}\frac{\pi^{2j}(2 - 2^{2j})B_{2j}}{(n-1)(n-j+1)2^{2j}a^{2j}b^{2j}} + \frac{2\pi b^2 c_n}{a^{2n+1}} + \frac{2\pi c_{n+1}}{a^{2n+3}} + \frac{4(-1)^n b^2}{a^{2n+1}}\Delta_n(ab) \\ &\quad - \frac{4(-1)^n\Delta_{n+1}(ab)}{a^{2n+3}} + \frac{2c(2^{2n+2} - 2)}{2^{2n+2}}(-1)^{n+1}\left(\frac{\pi}{a}\right)^{2n+3}B_{2n+2} \\ &\quad + 2b^2(-1)^{n+1}\left(\frac{\pi}{a}\right)^{2n+1}\frac{(2 - 2^{2n})}{2^{2n}}B_{2n}(c - 1)\end{aligned}$$

where, as before, Δ_n, B_n and c_n are given by (2.5), (3.1) and (3.20) respectively. In particular $b \to I_n(b, a, ; c)$ admits an explicit holomorphic continuation to the domain $U_{\pi/2a}$ (see definition (2.3))

For $n = 1$ equation (2.13) reduces to

$$\Delta_1(0) = \tfrac{1}{2}(-\pi^3 \log \pi) - \tfrac{7}{12}\pi^3 \log 2 + \tfrac{1}{4}\pi^3 - 3\pi^3\zeta'(-1). \tag{4.3}$$

By (3.20) and (4.3) we get

$$c_1 = -\pi^2[\tfrac{5}{6} \log \pi + \log 2 - \tfrac{7}{12} + 4\zeta'(-1)] \tag{4.4}$$

using $B_2 = \frac{1}{6}$. As $B_0 = 1$ and $c_0 = 2\log\pi$ by (3.21), one has from (2.7) and (4.4) the following very special case of theorem (4.2). (taking $n = 0$ there).

Corollary (4.5). For $a, b, > 0$

$$\begin{aligned}\pi I_0(b, a, ; 1) &\stackrel{\text{def}}{=} \pi \int_{-\infty}^{\infty} (\operatorname{sech}^2 at)(b^2 + t^2)\{\log(b^2 + t^2) - 1\}\, dt \\ &= \frac{4\pi b^2}{a} \log\frac{\pi}{a} - \frac{1}{3}\left(\frac{\pi}{a}\right)^3 \log a - \frac{1}{6}\left(\frac{\pi}{a}\right)^3 + 4\left(\frac{\pi}{a}\right)^3 \log\Gamma\left(\frac{ab}{\pi} + \frac{1}{2}\right) \\ &\quad - 4\left(\frac{\pi}{a}\right)^3\left(\frac{ab}{\pi} - \frac{1}{2}\right)\log 2\pi + 4\left(\frac{\pi}{a}\right)^3\left(\frac{ab}{\pi} - \frac{1}{2}\right)\left(\frac{ab}{\pi} + \frac{1}{2}\right) \\ &\quad - 8\left(\frac{\pi}{a}\right)^3 \log\Gamma_2\left(\frac{ab}{\pi} + \frac{1}{2}\right) - 2\left(\frac{\pi}{a}\right)^3\left[4\zeta'(-1) + \frac{5}{6}\log\pi + \log 2 - \frac{7}{12}\right]\end{aligned}$$

with Γ_2 defined by (2.1).

Note that since $\Gamma(1) = \Gamma_2(1) = 1$ we obtain a proof of (1.4) by specializing $b = \frac{1}{2}$, $a = \pi$ in corollary (4.5).

As in theorem (4.2) the functions $b \to J_n(b, a)$, defined initially for $b > 0$, admit an explicit holomorphic continuation via theorem (3.19).

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